Exercise 20

In Exercises 15-20, you are asked to solve the wave equation (1) subject to the boundary conditions (3), (4), and the initial conditions (5), (6), for the given data. [Hint: Use (10) and the remark that follows it.]

$$f(x) = \frac{1}{2}\sin\frac{2\pi x}{L}\cos\frac{2\pi x}{L}, \quad g(x) = 0$$

Solution

The general solution to the wave equation on a finite interval with fixed ends and arbitrary initial shape and zero velocity,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ -\infty < t < \infty$$

$$u(x,0) = \frac{1}{2}\sin\frac{2\pi x}{L}\cos\frac{2\pi x}{L} = \frac{1}{2}\left(\frac{1}{2}\sin\frac{4\pi x}{L}\right) = \frac{1}{4}\sin\frac{4\pi x}{L}$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

$$u(0,t) = 0$$

$$u(L,t) = 0,$$

is (to be derived in later chapters)

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}.$$

To determine the constants A_n, set $t = 0$ and substitute the given function for $u(x,0)$.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$\frac{1}{4}\sin\frac{4\pi x}{L} = A_1\sin\frac{\pi x}{L} + A_2\sin\frac{2\pi x}{L} + A_3\sin\frac{3\pi x}{L} + \cdots$$

Then match the coefficients on both sides.

$$A_1 = 0$$

$$A_2 = 0$$

$$A_4 = \frac{1}{4}$$

$$\vdots$$

$$A_n = 0, \quad n \neq 4$$

Therefore, the general solution that satisfies the initial conditions is

$$\begin{aligned} u(x,t) &= A_4 \sin\frac{4\pi x}{L}\cos\frac{4\pi ct}{L} \\ &= \frac{1}{4}\sin\frac{4\pi x}{L}\cos\frac{4\pi ct}{L}. \end{aligned}$$